# Infinite Series

## Exercise Set 9.1

**1.** **(a)** $\dfrac{1}{3^{n-1}}$ **(b)** $\dfrac{(-1)^{n-1}}{3^{n-1}}$ **(c)** $\dfrac{2n-1}{2n}$ **(d)** $\dfrac{n^2}{\pi^{1/(n+1)}}$

**2.** **(a)** $(-r)^{n-1}$; $(-r)^n$ **(b)** $-(-r)^n$; $(-1)^n r^{n+1}$

**3.** **(a)** $2, 0, 2, 0$ **(b)** $1, -1, 1, -1$ **(c)** $2(1+(-1)^n)$; $2+2\cos n\pi$

**4.** **(a)** $(2n)!$ **(b)** $(2n-1)!$

**5.** **(a)** No; $f(n)$ oscillates between $\pm 1$ and 0. **(b)** $-1, +1, -1, +1, -1$ **(c)** No, it oscillates between $+1$ and $-1$.

**6.** If $n$ is an integer then $f(2n+1) = 0$.

**(a)** $0, 0, 0, 0, 0$ **(b)** $b_n = 0$ for all $n$, so the sequence converges to 0. **(c)** No, it oscillates between $\pm 1$ and 0.

**7.** $1/3, 2/4, 3/5, 4/6, 5/7, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n}{n+2} = 1$, converges.

**8.** $1/3, 4/5, 9/7, 16/9, 25/11, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n^2}{2n+1} = +\infty$, diverges.

**9.** $2, 2, 2, 2, 2, \ldots$; $\lim\limits_{n\to+\infty} 2 = 2$, converges.

**10.** $\ln 1, \ln\dfrac{1}{2}, \ln\dfrac{1}{3}, \ln\dfrac{1}{4}, \ln\dfrac{1}{5}, \ldots$; $\lim\limits_{n\to+\infty} \ln(1/n) = -\infty$, diverges.

**11.** $\dfrac{\ln 1}{1}, \dfrac{\ln 2}{2}, \dfrac{\ln 3}{3}, \dfrac{\ln 4}{4}, \dfrac{\ln 5}{5}, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{\ln n}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ $\left(\text{apply L'Hôpital's Rule to } \dfrac{\ln x}{x}\right)$, converges.

**12.** $\sin\pi, 2\sin(\pi/2), 3\sin(\pi/3), 4\sin(\pi/4), 5\sin(\pi/5), \ldots$; $\lim\limits_{n\to+\infty} n\sin(\pi/n) = \lim\limits_{n\to+\infty} \dfrac{\sin(\pi/n)}{1/n}$; but using L'Hospital's rule, $\lim\limits_{x\to+\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \pi$, so the sequence also converges to $\pi$.

**13.** $0, 2, 0, 2, 0, \ldots$; diverges.

**14.** $1, -1/4, 1/9, -1/16, 1/25, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$, converges.

**15.** $-1, 16/9, -54/28, 128/65, -250/126, \ldots$; diverges because odd-numbered terms approach $-2$, even-numbered terms approach 2.





**16.** $1/2, 2/4, 3/8, 4/16, 5/32, \ldots$; using L'Hospital's rule, $\lim\limits_{x\to+\infty} \dfrac{x}{2^x} = \lim\limits_{x\to+\infty} \dfrac{1}{2^x \ln 2} = 0$, so the sequence also converges to 0.

**17.** $6/2, 12/8, 20/18, 30/32, 42/50, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{1}{2}(1+1/n)(1+2/n) = 1/2$, converges.

**18.** $\pi/4, \pi^2/4^2, \pi^3/4^3, \pi^4/4^4, \pi^5/4^5, \ldots$; $\lim\limits_{n\to+\infty} (\pi/4)^n = 0$, converges.

**19.** $e^{-1}, 4e^{-2}, 9e^{-3}, 16e^{-4}, 25e^{-5}, \ldots$; using L'Hospital's rule, $\lim\limits_{x\to+\infty} x^2e^{-x} = \lim\limits_{x\to+\infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2}{e^x} = 0$, so $\lim\limits_{n\to+\infty} n^2e^{-n} = 0$, converges.

**20.** $1, \sqrt{10}-2, \sqrt{18}-3, \sqrt{28}-4, \sqrt{40}-5, \ldots$; $\lim\limits_{n\to+\infty} (\sqrt{n^2+3n}-n) = \lim\limits_{n\to+\infty} \dfrac{3n}{\sqrt{n^2+3n}+n} = \lim\limits_{n\to+\infty} \dfrac{3}{\sqrt{1+3/n}+1} = \dfrac{3}{2}$, converges.

**21.** $2, (5/3)^2, (6/4)^3, (7/5)^4, (8/6)^5, \ldots$; let $y = \left[\dfrac{x+3}{x+1}\right]^x$, converges because $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln\dfrac{x+3}{x+1}}{1/x} = \lim\limits_{x\to+\infty} \dfrac{2x^2}{(x+1)(x+3)} = 2$, so $\lim\limits_{n\to+\infty} \left[\dfrac{n+3}{n+1}\right]^n = e^2$.

**22.** $-1, 0, (1/3)^3, (2/4)^4, (3/5)^5, \ldots$; let $y = (1-2/x)^x$, converges because $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln(1-2/x)}{1/x} = \lim\limits_{x\to+\infty} \dfrac{-2}{1-2/x} = -2$, $\lim\limits_{n\to+\infty} (1-2/n)^n = \lim\limits_{x\to+\infty} y = e^{-2}$.

**23.** $\left\{\dfrac{2n-1}{2n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{2n-1}{2n} = 1$, converges.

**24.** $\left\{\dfrac{n-1}{n^2}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{n-1}{n^2} = 0$, converges.

**25.** $\left\{(-1)^{n-1}\dfrac{1}{3^n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n-1}}{3^n} = 0$, converges.

**26.** $\{(-1)^n n\}_{n=1}^{+\infty}$; diverges because odd-numbered terms tend toward $-\infty$, even-numbered terms tend toward $+\infty$.

27. $\left\{(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}_{n=1}^{+\infty}$; the sequence converges to 0.

28. $\{3/2^{n-1}\}_{n=1}^{+\infty}$; $\lim_{n\to+\infty} 3/2^{n-1}=0$, converges.

29. $\{\sqrt{n+1}-\sqrt{n+2}\}_{n=1}^{+\infty}$; converges because $\lim_{n\to+\infty}(\sqrt{n+1}-\sqrt{n+2})=\lim_{n\to+\infty}\frac{(n+1)-(n+2)}{\sqrt{n+1}+\sqrt{n+2}}=$
$=\lim_{n\to+\infty}\frac{-1}{\sqrt{n+1}+\sqrt{n+2}}=0.$

30. $\{(-1)^{n+1}/3^{n+4}\}_{n=1}^{+\infty}$; $\lim_{n\to+\infty}(-1)^{n+1}/3^{n+4}=0$, converges.

31. True; a function whose domain is a set of integers.



32. False, e.g. $a_n=1-n, b_n=n-1$.

33. False, e.g. $a_n=(-1)^n$.

34. True.

35. Let $a_n=0, b_n=\frac{\sin^2 n}{n}, c_n=\frac{1}{n}$; then $a_n\le b_n\le c_n$, $\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}c_n=0$, so $\lim_{n\to+\infty}b_n=0$.

36. Let $a_n=0, b_n=\left(\frac{1+n}{2n}\right)^n, c_n=\left(\frac{3}{4}\right)^n$; then (for $n\ge 2$), $a_n\le b_n\le\left(\frac{n/2+n}{2n}\right)^n=c_n$,
$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}c_n=0$, so $\lim_{n\to+\infty}b_n=0$.

37. $a_n=\begin{cases}+1 & k \text{ even}\\ -1 & k \text{ odd}\end{cases}$ oscillates; there is no limit point which attracts all of the $a_n$. $b_n=\cos n$; the terms lie all over the interval $[-1,1]$ without any limit.

38. **(a)** No, because given $N>0$, all values of $f(x)$ are greater than $N$ provided $x$ is close enough to zero. But certainly the terms $1/n$ will be arbitrarily close to zero, and when so then $f(1/n)>N$, so $f(1/n)$ cannot converge.

**(b)** $f(x)=\sin(\pi/x)$. Then $f=0$ when $x=1/n$ and $f\ne 0$ otherwise; indeed, the values of $f$ are located all over the interval $[-1,1]$.

39. **(a)** $1,2,1,4,1,6$ **(b)** $a_n=\begin{cases}n, & n \text{ odd}\\ 1/2^n, & n \text{ even}\end{cases}$ **(c)** $a_n=\begin{cases}1/n, & n \text{ odd}\\ 1/(n+1), & n \text{ even}\end{cases}$

**(d)** In part (a) the sequence diverges, since the even terms diverge to $+\infty$ and the odd terms equal 1; in part (b) the sequence diverges, since the odd terms diverge to $+\infty$ and the even terms tend to zero; in part (c) $\lim_{n\to+\infty}a_n=0$.

40. The even terms are zero, so the odd terms must converge to zero, and this is true if and only if $\lim_{n\to+\infty}b^n=0$, or $0<b<1$ ($b$ is required to be positive).

41. $\lim_{n\to+\infty}x_{n+1}=\frac{1}{2}\lim_{n\to+\infty}\left(x_n+\frac{a}{x_n}\right)$ or $L=\frac{1}{2}\left(L+\frac{a}{L}\right)$, $2L^2-L^2-a=0$, $L=\sqrt{a}$ (we reject $-\sqrt{a}$ because $x_n>0$, thus $L\ge 0$).

42. **(a)** $a_{n+1}=\sqrt{6+a_n}$.

**(b)** $\lim_{n\to+\infty}a_{n+1}=\lim_{n\to+\infty}\sqrt{6+a_n}$, $L=\sqrt{6+L}$, $L^2-L-6=0$, $(L-3)(L+2)=0$, $L=-2$ (reject, because the terms in the sequence are positive) or $L=3$; $\lim_{n\to+\infty}a_n=3$.

43. **(a)** $a_1=(0.5)^2, a_2=a_1^2=(0.5)^4,\ldots,a_n=(0.5)^{2^n}$.

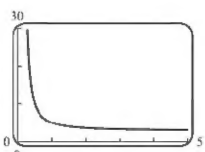
**(c)** $\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}e^{2^n\ln(0.5)}=0$, since $\ln(0.5)<0$.

**(d)** Replace 0.5 in part (a) with $a_0$; then the sequence converges for $-1\le a_0\le 1$, because if $a_0=\pm 1$, then $a_n=1$ for $n\ge 1$; if $a_0=0$ then $a_n=0$ for $n\ge 1$; and if $0<|a_0|<1$ then $a_1=a_0^2>0$ and $\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}e^{2^{n-1}\ln a_1}=0$ since $0<a_1<1$. This same argument proves divergence to $+\infty$ for $|a|>1$ since then $\ln a_1>0$.

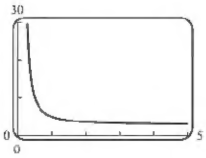
44. $f(0.2)=0.4$, $f(0.4)=0.8$, $f(0.8)=0.6$, $f(0.6)=0.2$ and then the cycle repeats, so the sequence does not converge.



45. **(a)**




**(b)** Let $y=(2^x+3^x)^{1/x}$, $\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\frac{\ln(2^x+3^x)}{x}=\lim_{x\to+\infty}\frac{2^x\ln 2+3^x\ln 3}{2^x+3^x}=\lim_{x\to+\infty}\frac{(2/3)^x\ln 2+\ln 3}{(2/3)^x+1}=$
$\ln 3$, so $\lim_{n\to+\infty}(2^n+3^n)^{1/n}=e^{\ln 3}=3$. Alternate proof: $3=(3^n)^{1/n}<(2^n+3^n)^{1/n}<(2\cdot 3^n)^{1/n}=3\cdot 2^{1/n}$. Then

**45.** **(a)** 


**(b)** Let $y = (2^x + 3^x)^{1/x}$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln(2^x + 3^x)}{x} = \lim\limits_{x\to+\infty} \dfrac{2^x \ln 2 + 3^x \ln 3}{2^x + 3^x} = \lim\limits_{x\to+\infty} \dfrac{(2/3)^x \ln 2 + \ln 3}{(2/3)^x + 1} =$ $\ln 3$, so $\lim\limits_{n\to+\infty} (2^n + 3^n)^{1/n} = e^{\ln 3} = 3$. Alternate proof: $3 = (3^n)^{1/n} < (2^n + 3^n)^{1/n} < (2 \cdot 3^n)^{1/n} = 3 \cdot 2^{1/n}$. Then apply the Squeezing Theorem.

**46.** Let $f(x) = 1/(1 + x)$, $0 \le x \le 1$. Take $\Delta x_k = 1/n$ and $x_k^* = k/n$ then $a_n = \displaystyle\sum_{k=1}^{n} \frac{1}{1 + (k/n)}(1/n) = \sum_{k=1}^{n} \frac{1}{1 + x_k^*}\Delta x_k$ so $\lim\limits_{n\to+\infty} a_n = \displaystyle\int_0^1 \frac{1}{1+x}dx = \ln(1 + x)\Big]_0^1 = \ln 2$.

**47.** **(a)** If $n \ge 1$, then $a_{n+2} = a_{n+1} + a_n$, so $\dfrac{a_{n+2}}{a_{n+1}} = 1 + \dfrac{a_n}{a_{n+1}}$.

**(c)** With $L = \lim\limits_{n\to+\infty} (a_{n+2}/a_{n+1}) = \lim\limits_{n\to+\infty} (a_{n+1}/a_n)$, $L = 1 + 1/L$, $L^2 - L - 1 = 0$, $L = (1 \pm \sqrt{5})/2$, so $L = (1 + \sqrt{5})/2$ because the limit cannot be negative.

**48.** $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \epsilon$ if $n > 1/\epsilon$;

**(a)** $1/\epsilon = 1/0.5 = 2$, $N = 3$. **(b)** $1/\epsilon = 1/0.1 = 10$, $N = 11$. **(c)** $1/\epsilon = 1/0.001 = 1000$, $N = 1001$.

**49.** $\left|\dfrac{n}{n+1} - 1\right| = \dfrac{1}{n+1} < \epsilon$ if $n + 1 > 1/\epsilon$, $n > 1/\epsilon - 1$;

**(a)** $1/\epsilon - 1 = 1/0.25 - 1 = 3$, $N = 4$. **(b)** $1/\epsilon - 1 = 1/0.1 - 1 = 9$, $N = 10$. **(c)** $1/\epsilon - 1 = 1/0.001 - 1 = 999$, $N = 1000$

**50.** **(a)** $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \epsilon$ if $n > 1/\epsilon$, choose any $N > 1/\epsilon$.

**(b)** $\left|\dfrac{n}{n+1} - 1\right| = \dfrac{1}{n+1} < \epsilon$ if $n > 1/\epsilon - 1$, choose any $N > 1/\epsilon - 1$.

## Exercise Set 9.2

**1.** $a_{n+1} - a_n = \dfrac{1}{n+1} - \dfrac{1}{n} = -\dfrac{1}{n(n+1)} < 0$ for $n \ge 1$, so strictly decreasing.

**2.** $a_{n+1} - a_n = \left(1 - \dfrac{1}{n+1}\right) - \left(1 - \dfrac{1}{n}\right) = \dfrac{1}{n(n+1)} > 0$ for $n \ge 1$, so strictly increasing.

**3.** $a_{n+1} - a_n = \dfrac{n+1}{2n+3} - \dfrac{n}{2n+1} = \dfrac{1}{(2n+1)(2n+3)} > 0$ for $n \ge 1$, so strictly increasing.

**4.** $a_{n+1} - a_n = \dfrac{n+1}{4n+3} - \dfrac{n}{4n-1} = -\dfrac{1}{(4n-1)(4n+3)} < 0$ for $n \ge 1$, so strictly decreasing.



**5.** $a_{n+1} - a_n = (n + 1 - 2^{n+1}) - (n - 2^n) = 1 - 2^n < 0$ for $n \ge 1$, so strictly decreasing.

**6.** $a_{n+1} - a_n = [(n+1) - (n+1)^2] - (n - n^2) = -2n < 0$ for $n \ge 1$, so strictly decreasing.

**7.** $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)/(2n+3)}{n/(2n+1)} = \dfrac{(n+1)(2n+1)}{n(2n+3)} = \dfrac{2n^2+3n+1}{2n^2+3n} > 1$ for $n \ge 1$, so strictly increasing.

**8.** $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{1+2^{n+1}} \cdot \dfrac{1+2^n}{2^n} = \dfrac{2+2^{n+1}}{1+2^{n+1}} = 1 + \dfrac{1}{1+2^{n+1}} > 1$ for $n \ge 1$, so strictly increasing.

**9.** $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)e^{-(n+1)}}{ne^{-n}} = (1 + 1/n)e^{-1} < 1$ for $n \ge 1$, so strictly decreasing.

**10.** $\dfrac{a_{n+1}}{a_n} = \dfrac{10^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{10^n} = \dfrac{10}{(2n+2)(2n+1)} < 1$ for $n \ge 1$, so strictly decreasing.

**11.** $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \dfrac{(n+1)^n}{n^n} = (1 + 1/n)^n > 1$ for $n \ge 1$, so strictly increasing.

**12.** $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}}{2^{(n+1)^2}} \cdot \dfrac{2^{n^2}}{5^n} = \dfrac{5}{2^{2n+1}} < 1$ for $n \ge 1$, so strictly decreasing.

**13.** True by definition.

**14.** False; either $a_{n+1} \le a_n$ always or else $a_{n+1} \ge a_n$ always.

**15.** False, e.g. $a_n = (-1)^n$.

**16.** False; such a sequence could decrease until $a_{300}$, e.g.

**17.** $f(x) = x/(2x+1)$, $f'(x) = 1/(2x+1)^2 > 0$ for $x \geq 1$, so strictly increasing.

**18.** $f(x) = \dfrac{\ln(x+2)}{x+2}$, $f'(x) = \dfrac{1-\ln(x+2)}{(x+2)^2} < 0$ for $x \geq 1$, so strictly decreasing.

**19.** $f(x) = \tan^{-1} x$, $f'(x) = 1/(1+x^2) > 0$ for $x \geq 1$, so strictly increasing.

**20.** $f(x) = xe^{-2x}$, $f'(x) = (1-2x)e^{-2x} < 0$ for $x \geq 1$, so strictly decreasing.

**21.** $f(x) = 2x^2 - 7x$, $f'(x) = 4x - 7 > 0$ for $x \geq 2$, so eventually strictly increasing.

**22.** $f(x) = \dfrac{x}{x^2+10}$, $f'(x) = \dfrac{10-x^2}{(x^2+10)^2} < 0$ for $x \geq 4$, so eventually strictly decreasing.

**23.** $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{3^{n+1}} \cdot \dfrac{3^n}{n!} = \dfrac{n+1}{3} > 1$ for $n \geq 3$, so eventually strictly increasing.

**24.** $f(x) = x^5 e^{-x}$, $f'(x) = x^4(5-x)e^{-x} < 0$ for $x \geq 6$, so eventually strictly decreasing.

**25.** Yes: a monotone sequence is increasing or decreasing; if it is increasing, then it is increasing and bounded above, so by Theorem 9.2.3 it converges; if decreasing, then use Theorem 9.2.4. The limit lies in the interval $[1, 2]$.

**26.** Such a sequence may converge, in which case, by the argument in part (a), its limit is $\leq 2$. If the sequence is also increasing then it will converge. But convergence may not happen: for example, the sequence $\{-n\}_{n=1}^{+\infty}$ diverges.



**27. (a)** $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$.

**(b)** $a_1 = \sqrt{2} < 2$ so $a_2 = \sqrt{2+a_1} < \sqrt{2+2} = 2$, $a_3 = \sqrt{2+a_2} < \sqrt{2+2} = 2$, and so on indefinitely.

**(c)** $a_{n+1}^2 - a_n^2 = (2+a_n) - a_n^2 = 2 + a_n - a_n^2 = (2-a_n)(1+a_n)$.

**(d)** $a_n > 0$ and, from part (b), $a_n < 2$ so $2 - a_n > 0$ and $1 + a_n > 0$ thus, from part (c), $a_{n+1}^2 - a_n^2 > 0$, $a_{n+1} - a_n > 0$, $a_{n+1} > a_n$; $\{a_n\}$ is a strictly increasing sequence.

**(e)** The sequence is increasing and has 2 as an upper bound so it must converge to a limit $L$, $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \sqrt{2+a_n}$, $L = \sqrt{2+L}$, $L^2 - L - 2 = 0$, $(L-2)(L+1) = 0$, thus $\lim\limits_{n\to+\infty} a_n = 2$.

**28. (a)** If $f(x) = \dfrac{1}{2}(x + 3/x)$, then $f'(x) = (x^2-3)/(2x^2)$ and $f'(x) = 0$ for $x = \sqrt{3}$; the minimum value of $f(x)$ for $x > 0$ is $f(\sqrt{3}) = \sqrt{3}$. Thus $f(x) \geq \sqrt{3}$ for $x > 0$ and hence $a_n \geq \sqrt{3}$ for $n \geq 2$.

**(b)** $a_{n+1} - a_n = (3 - a_n^2)/(2a_n) \leq 0$ for $n \geq 2$ since $a_n \geq \sqrt{3}$ for $n \geq 2$; $\{a_n\}$ is eventually decreasing.

**(c)** $\sqrt{3}$ is a lower bound for $a_n$ so $\{a_n\}$ converges; $\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \dfrac{1}{2}(a_n + 3/a_n)$, $L = \dfrac{1}{2}(L + 3/L)$, $L^2 - 3 = 0$, $L = \sqrt{3}$.

**29. (a)** $x_1 = 60$, $x_2 = \dfrac{1500}{7} \approx 214.3$, $x_3 = \dfrac{3750}{13} \approx 288.5$, $x_4 = \dfrac{75000}{251} \approx 298.8$.

**(b)** We can see that $x_{n+1} = \dfrac{RK}{K/x_n + (R-1)} = \dfrac{10\cdot 300}{300/x_n + 9}$; if $0 < x_n$ then clearly $0 < x_{n+1}$. Also, if $x_n < 300$, then $x_{n+1} = \dfrac{10 \cdot 300}{300/x_n + 9} < \dfrac{10 \cdot 300}{300/300 + 9} = 300$, so the conclusion is valid.

**(c)** $\dfrac{x_{n+1}}{x_n} = \dfrac{RK}{K + (R-1)x_n} = \dfrac{10 \cdot 300}{300 + 9x_n} > \dfrac{10 \cdot 300}{300 + 9 \cdot 300} = 1$, because $x_n < 300$. So $x_n$ is increasing.

**(d)** $x_n$ is increasing and bounded above, so it is convergent. The limit can be found by letting $L = \dfrac{RKL}{K + (R-1)L}$, this gives us $L = K = 300$. (The other root, $L = 0$ can be ruled out by the increasing property of the sequence.)

**30. (a)** Again, $x_{n+1} = \dfrac{RK}{K/x_n + (R-1)}$, so if $x_n > K$, then $x_{n+1} = \dfrac{RK}{K/x_n + (R-1)} > \dfrac{RK}{K/K + (R-1)} = K$, so the conclusion is valid (we only used $R > 1$ and $K > 0$).

**(b)** $\dfrac{x_{n+1}}{x_n} = \dfrac{RK}{K + (R-1)x_n} < \dfrac{RK}{K + (R-1)K} = 1$, because $x_n > K$. So $x_n$ is decreasing.

**(c)** $x_n$ is decreasing and bounded below, so it is convergent. The limit can be found by letting $L = \dfrac{RKL}{K + (R-1)L}$, this gives us $L = K$. (The other root, $L = 0$ can be ruled out by the fact that $x_n > K$.)

**31. (a)** $a_{n+1} = \dfrac{|x|^{n+1}}{(n+1)!} = \dfrac{|x|}{n+1}\dfrac{|x|^n}{n!} = \dfrac{|x|}{n+1}a_n$.

**(b)** $a_{n+1}/a_n = |x|/(n+1) < 1$ if $n > |x| - 1$.

**(c)** From part (b) the sequence is eventually decreasing, and it is bounded below by 0, so by Theorem 9.2.4 it converges.



**32. (a)** The altitudes of the rectangles are $\ln k$ for $k = 2$ to $n$, and their bases all have length 1 so the sum of their areas is $\ln 2 + \ln 3 + \ldots + \ln n = \ln(2 \cdot 3 \cdot \ldots \cdot n) = \ln n!$. The area under the curve $y = \ln x$ for $x$ in

**32.** **(a)** The altitudes of the rectangles are $\ln k$ for $k = 2$ to $n$, and their bases all have length 1 so the sum of their areas is $\ln 2 + \ln 3 + \ldots + \ln n = \ln(2 \cdot 3 \cdot \ldots \cdot n) = \ln n!$. The area under the curve $y = \ln x$ for $x$ in the interval $[1, n]$ is $\int_1^n \ln x\, dx$, and $\int_1^{n+1} \ln x\, dx$ is the area for $x$ in the interval $[1, n+1]$ so, from the figure, $\int_1^n \ln x\, dx < \ln n! < \int_1^{n+1} \ln x\, dx$.

**(b)** $\int_1^n \ln x\, dx = (x\ln x - x)\Big]_1^n = n\ln n - n + 1$ and $\int_1^{n+1} \ln x\, dx = (n+1)\ln(n+1) - n$, so from part (a), $n\ln n - n + 1 < \ln n! < (n+1)\ln(n+1) - n$, $e^{n\ln n - n + 1} < n! < e^{(n+1)\ln(n+1) - n}$, $e^{n\ln n}e^{1-n} < n! < e^{(n+1)\ln(n+1)}e^{-n}$, $\dfrac{n^n}{e^{n-1}} < n! < \dfrac{(n+1)^{n+1}}{e^n}$.

**(c)** From part (b), $\left[\dfrac{n^n}{e^{n-1}}\right]^{1/n} < \sqrt[n]{n!} < \left[\dfrac{(n+1)^{n+1}}{e^n}\right]^{1/n}$, $\dfrac{n}{e^{1-1/n}} < \sqrt[n]{n!} < \dfrac{(n+1)^{1+1/n}}{e}$, $\dfrac{1}{e^{1-1/n}} < \dfrac{\sqrt[n]{n!}}{n} < \dfrac{(1+1/n)(n+1)^{1/n}}{e}$, but $\dfrac{1}{e^{1-1/n}} \to \dfrac{1}{e}$ and $\dfrac{(1+1/n)(n+1)^{1/n}}{e} \to \dfrac{1}{e}$ as $n \to +\infty$ (why?), so $\lim_{n\to+\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

**33.** $n! > \dfrac{n^n}{e^{n-1}}$, $\sqrt[n]{n!} > \dfrac{n}{e^{1-1/n}}$, $\lim_{n\to+\infty} \dfrac{n}{e^{1-1/n}} = +\infty$, so $\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$.

## Exercise Set 9.3

**1.** **(a)** $s_1 = 2$, $s_2 = 12/5$, $s_3 = \dfrac{62}{25}$, $s_4 = \dfrac{312}{125}$ $s_n = \dfrac{2 - 2(1/5)^n}{1 - 1/5} = \dfrac{5}{2} - \dfrac{5}{2}(1/5)^n$, $\lim_{n\to+\infty} s_n = \dfrac{5}{2}$, converges.

**(b)** $s_1 = \dfrac{1}{4}$, $s_2 = \dfrac{3}{4}$, $s_3 = \dfrac{7}{4}$, $s_4 = \dfrac{15}{4}$ $s_n = \dfrac{(1/4) - (1/4)2^n}{1 - 2} = -\dfrac{1}{4} + \dfrac{1}{4}(2^n)$, $\lim_{n\to+\infty} s_n = +\infty$, diverges.

**(c)** $\dfrac{1}{(k+1)(k+2)} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$, $s_1 = \dfrac{1}{6}$, $s_2 = \dfrac{1}{4}$, $s_3 = \dfrac{3}{10}$, $s_4 = \dfrac{1}{3}$; $s_n = \dfrac{1}{2} - \dfrac{1}{n+2}$, $\lim_{n\to+\infty} s_n = \dfrac{1}{2}$, converges.

**2.** **(a)** $s_1 = 1/4$, $s_2 = 5/16$, $s_3 = 21/64$, $s_4 = 85/256$, $s_n = \dfrac{1}{4}\left(1 + \dfrac{1}{4} + \ldots + \left(\dfrac{1}{4}\right)^{n-1}\right) = \dfrac{1}{4}\dfrac{1 - (1/4)^n}{1 - 1/4} = \dfrac{1}{3}\left(1 - \left(\dfrac{1}{4}\right)^n\right)$; $\lim_{n\to+\infty} s_n = \dfrac{1}{3}$.

**(b)** $s_1 = 1$, $s_2 = 5$, $s_3 = 21$, $s_4 = 85$; $s_n = \dfrac{4^n - 1}{3}$, diverges.

**(c)** $s_1 = 1/20$, $s_2 = 1/12$, $s_3 = 3/28$, $s_4 = 1/8$; $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+3} - \frac{1}{k+4}\right) = \frac{1}{4} - \frac{1}{n+4}$, $\lim_{n\to+\infty} s_n = 1/4$.

**3.** Geometric, $a = 1$, $r = -3/4$, $|r| = 3/4 < 1$, series converges, sum $= \dfrac{1}{1 - (-3/4)} = 4/7$.

**4.** Geometric, $a = (2/3)^3$, $r = 2/3$, $|r| = 2/3 < 1$, series converges, sum $= \dfrac{(2/3)^3}{1 - 2/3} = 8/9$.

**5.** Geometric, $a = 7$, $r = -1/6$, $|r| = 1/6 < 1$, series converges, sum $= \dfrac{7}{1 + 1/6} = 6$.

**6.** Geometric, $r = -3/2$, $|r| = 3/2 \geq 1$, diverges.



**7.** $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \frac{1}{3} - \frac{1}{n+3}$, $\lim_{n\to+\infty} s_n = 1/3$, series converges by definition, sum $= 1/3$.

**8.** $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{1}{2} - \frac{1}{2^{n+1}}$, $\lim_{n\to+\infty} s_n = 1/2$, series converges by definition, sum $= 1/2$.

**9.** $s_n = \displaystyle\sum_{k=1}^{n}\left(\frac{1/3}{3k-1} - \frac{1/3}{3k+2}\right) = \frac{1}{6} - \frac{1/3}{3n+2}$, $\lim_{n\to+\infty} s_n = 1/6$, series converges by definition, sum $= 1/6$.

**10.** $s_n = \displaystyle\sum_{k=2}^{n+1}\left[\frac{1/2}{k-1} - \frac{1/2}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=2}^{n+1}\frac{1}{k+1}\right] = \frac{1}{2}\left[\sum_{k=2}^{n+1}\frac{1}{k-1} - \sum_{k=4}^{n+3}\frac{1}{k-1}\right] =$
$= \dfrac{1}{2}\left[1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right]$; $\lim_{n\to+\infty} s_n = \dfrac{3}{4}$, series converges by definition, sum $= 3/4$.

**11.** $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k-2} = \sum_{k=1}^{\infty} 1/k$, the harmonic series, so the series diverges.

**12.** Geometric, $a = (e/\pi)^4$, $r = e/\pi$, $|r| = e/\pi < 1$, series converges, sum $= \dfrac{(e/\pi)^4}{1 - e/\pi} = \dfrac{e^4}{\pi^3(\pi - e)}$.

**13.** $\displaystyle\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}} = \sum_{k=1}^{\infty} 64\left(\frac{4}{7}\right)^{k-1}$; geometric, $a = 64$, $r = 4/7$, $|r| = 4/7 < 1$, series converges, sum $= \dfrac{64}{1 - 4/7} = 448/3$.

**14.** Geometric, $a = 125$, $r = 125/7$, $|r| = 125/7 \geq 1$, diverges.

**15.** **(a)** Exercise 5 **(b)** Exercise 3 **(c)** Exercise 7 **(d)** Exercise 9

**16.** **(a)** Exercise 10 **(b)** Exercise 6 **(c)** Exercise 4 **(d)** Exercise 8

**17.** False; e.g. $a_n = 1/n$.

**18.** True, Theorem 9.3.3.

**19.** True.

**20.** True.

**21.** $0.9999\ldots = 0.9 + 0.09 + 0.009 + \ldots = \dfrac{0.9}{1-0.1} = 1.$

**22.** $0.4444\ldots = 0.4 + 0.04 + 0.004 + \ldots = \dfrac{0.4}{1-0.1} = 4/9.$

**23.** $5.373737\ldots = 5 + 0.37 + 0.0037 + 0.000037 + \ldots = 5 + \dfrac{0.37}{1-0.01} = 5 + 37/99 = 532/99.$

**24.** $0.451141414\ldots = 0.451 + 0.00014 + 0.0000014 + 0.000000014 + \ldots = 0.451 + \dfrac{0.00014}{1-0.01} = \dfrac{44663}{99000}.$

**25.** $0.a_1a_2\ldots a_n9999\ldots = 0.a_1a_2\ldots a_n + 0.9\,(10^{-n}) + 0.09\,(10^{-n}) + \ldots = 0.a_1a_2\ldots a_n + \dfrac{0.9\,(10^{-n})}{1-0.1} = 0.a_1a_2\ldots a_n + 10^{-n} = 0.a_1a_2\ldots(a_n+1) = 0.a_1a_2\ldots(a_n+1)\,0000\ldots$



**26.** The series converges to $1/(1-x)$ only if $-1 < x < 1$.

**27.** $d = 10 + 2\cdot\frac{3}{4}\cdot 10 + 2\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot 10 + 2\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot 10 + \ldots = 10 + 20\left(\frac{3}{4}\right) + 20\left(\frac{3}{4}\right)^2 + 20\left(\frac{3}{4}\right)^3 + \ldots = 10 + \dfrac{20(3/4)}{1-3/4} = 10 + 60 = 70$ meters.

**28.** Volume $= 1^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{4}\right)^3 + \ldots + \left(\frac{1}{2^n}\right)^3 + \ldots = 1 + \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \ldots + \left(\frac{1}{8}\right)^n + \ldots = \dfrac{1}{1-(1/8)} = 8/7.$

**29.** **(a)** $s_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1} = \ln\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}\right) = \ln\frac{1}{n+1} = -\ln(n+1)$, $\lim\limits_{n\to+\infty} s_n = -\infty$, series diverges.

**(b)** $\ln(1-1/k^2) = \ln\dfrac{k^2-1}{k^2} = \ln\dfrac{(k-1)(k+1)}{k^2} = \ln\dfrac{k-1}{k} + \ln\dfrac{k+1}{k} = \ln\dfrac{k-1}{k} - \ln\dfrac{k}{k+1}$, so
$s_n = \sum\limits_{k=2}^{n+1}\left[\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right] = \left(\ln\frac{1}{2} - \ln\frac{2}{3}\right) + \left(\ln\frac{2}{3} - \ln\frac{3}{4}\right) + \left(\ln\frac{3}{4} - \ln\frac{4}{5}\right) + \ldots + \left(\ln\frac{n}{n+1} - \ln\frac{n+1}{n+2}\right) = \ln\frac{1}{2} - \ln\frac{n+1}{n+2}$, and then $\lim\limits_{n\to+\infty} s_n = \ln\frac{1}{2} = -\ln 2$.

**30.** **(a)** $\sum\limits_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + \ldots = \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$ if $|-x| < 1$, $|x| < 1$, $-1 < x < 1$.

**(b)** $\sum\limits_{k=0}^{\infty}(x-3)^k = 1 + (x-3) + (x-3)^2 + \ldots = \dfrac{1}{1-(x-3)} = \dfrac{1}{4-x}$ if $|x-3| < 1$, $2 < x < 4$.

**(c)** $\sum\limits_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \ldots = \dfrac{1}{1-(-x^2)} = \dfrac{1}{1+x^2}$ if $|-x^2| < 1$, $|x| < 1$, $-1 < x < 1$.

**31.** **(a)** Geometric series, $a = x$, $r = -x^2$. Converges for $|-x^2| < 1$, $|x| < 1$; $S = \dfrac{x}{1-(-x^2)} = \dfrac{x}{1+x^2}$.

**(b)** Geometric series, $a = 1/x^2$, $r = 2/x$. Converges for $|2/x| < 1$, $|x| > 2$; $S = \dfrac{1/x^2}{1-2/x} = \dfrac{1}{x^2-2x}$.

**(c)** Geometric series, $a = e^{-x}$, $r = e^{-x}$. Converges for $|e^{-x}| < 1$, $e^{-x} < 1$, $e^x > 1$, $x > 0$; $S = \dfrac{e^{-x}}{1-e^{-x}} = \dfrac{1}{e^x - 1}$.

**32.** Geometric series, $a = \sin\, x$, $r = -\frac{1}{2}\sin x$. Converges for $|-\frac{1}{2}\sin x| < 1$, $|\sin x| < 2$, so converges for all values of $x$. $S = \dfrac{\sin x}{1+\frac{1}{2}\sin x} = \dfrac{2\sin x}{2+\sin x}$.

**33.** $a_2 = \frac{1}{2}a_1 + \frac{1}{2}$, $a_3 = \frac{1}{2}a_2 + \frac{1}{2} = \frac{1}{2^2}a_1 + \frac{1}{2^2} + \frac{1}{2}$, $a_4 = \frac{1}{2}a_3 + \frac{1}{2} = \frac{1}{2^3}a_1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2}$, $a_5 = \frac{1}{2}a_4 + \frac{1}{2} = \frac{1}{2^4}a_1 + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2}, \ldots, a_n = \frac{1}{2^{n-1}}a_1 + \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \ldots + \frac{1}{2}$, $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty}\frac{a_1}{2^{n-1}} + \sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 0 + \dfrac{1/2}{1-1/2} = 1.$

**34.** $\dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = \dfrac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}} = \dfrac{1}{\sqrt{k}} - \dfrac{1}{\sqrt{k+1}}$, $s_n = \sum\limits_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \dfrac{1}{\sqrt{n+1}}$; $\lim\limits_{n\to+\infty} s_n = 1$.





$(1/3-1/5) + (1/4-1/6) + \ldots + [1/n - 1/(n+2)] = (1 + 1/2 + 1/3 + \ldots + 1/n) -$
$+2)) = 3/2 - 1/(n+1) - 1/(n+2)$, $\lim\limits_{n\to+\infty} s_n = 3/2.$

**35.** $s_n = (1-1/3)+(1/2-1/4)+(1/3-1/5)+(1/4-1/6)+\ldots+[1/n-1/(n+2)] = (1+1/2+1/3+\ldots+1/n) - (1/3+1/4+1/5+\ldots+1/(n+2)) = 3/2 - 1/(n+1) - 1/(n+2)$, $\lim\limits_{n\to+\infty} s_n = 3/2$.

**36.** $s_n = \sum\limits_{k=1}^{n} \dfrac{1}{k(k+2)} = \sum\limits_{k=1}^{n}\left[\dfrac{1/2}{k} - \dfrac{1/2}{k+2}\right] = \dfrac{1}{2}\left[\sum\limits_{k=1}^{n}\dfrac{1}{k} - \sum\limits_{k=1}^{n}\dfrac{1}{k+2}\right] = \dfrac{1}{2}\left[\sum\limits_{k=1}^{n}\dfrac{1}{k} - \sum\limits_{k=3}^{n+2}\dfrac{1}{k}\right] =$
$= \dfrac{1}{2}\left[1+\dfrac{1}{2}-\dfrac{1}{n+1}-\dfrac{1}{n+2}\right]$; $\lim\limits_{n\to+\infty} s_n = \dfrac{3}{4}$.

**37.** $s_n = \sum\limits_{k=1}^{n}\dfrac{1}{(2k-1)(2k+1)} = \sum\limits_{k=1}^{n}\left[\dfrac{1/2}{2k-1} - \dfrac{1/2}{2k+1}\right] = \dfrac{1}{2}\left[\sum\limits_{k=1}^{n}\dfrac{1}{2k-1} - \sum\limits_{k=1}^{n}\dfrac{1}{2k+1}\right] =$
$= \dfrac{1}{2}\left[\sum\limits_{k=1}^{n}\dfrac{1}{2k-1} - \sum\limits_{k=2}^{n+1}\dfrac{1}{2k-1}\right] = \dfrac{1}{2}\left[1-\dfrac{1}{2n+1}\right]$; $\lim\limits_{n\to+\infty} s_n = \dfrac{1}{2}$.

**38.** $A_1 + A_2 + A_3 + \ldots = 1 + 1/2 + 1/4 + \ldots = \dfrac{1}{1-(1/2)} = 2$.

**39.** By inspection, $\dfrac{\theta}{2} - \dfrac{\theta}{4} + \dfrac{\theta}{8} - \dfrac{\theta}{16} + \ldots = \dfrac{\theta/2}{1-(-1/2)} = \theta/3$.

**40.** **(a)** Geometric; 18/5. **(b)** Geometric; diverges. **(c)** $\sum\limits_{k=1}^{\infty}\dfrac{1}{2}\left(\dfrac{1}{2k-1} - \dfrac{1}{2k+1}\right) = 1/2$.

## Exercise Set 9.4

**1.** **(a)** $\sum\limits_{k=1}^{\infty}\dfrac{1}{2^k} = \dfrac{1/2}{1-1/2} = 1$; $\sum\limits_{k=1}^{\infty}\dfrac{1}{4^k} = \dfrac{1/4}{1-1/4} = 1/3$; $\sum\limits_{k=1}^{\infty}\left(\dfrac{1}{2^k}+\dfrac{1}{4^k}\right) = 1 + 1/3 = 4/3$.

**(b)** $\sum\limits_{k=1}^{\infty}\dfrac{1}{5^k} = \dfrac{1/5}{1-1/5} = 1/4$; $\sum\limits_{k=1}^{\infty}\dfrac{1}{k(k+1)} = 1$, (Ex. 5, Section 9.3); $\sum\limits_{k=1}^{\infty}\left[\dfrac{1}{5^k} - \dfrac{1}{k(k+1)}\right] = 1/4 - 1 = -3/4$.

**2.** **(a)** $\sum\limits_{k=2}^{\infty}\dfrac{1}{k^2-1} = 3/4$ (Ex. 10, Section 9.3); $\sum\limits_{k=2}^{\infty}\dfrac{7}{10^{k-1}} = \dfrac{7/10}{1-1/10} = 7/9$; so $\sum\limits_{k=2}^{\infty}\left[\dfrac{1}{k^2-1} - \dfrac{7}{10^{k-1}}\right] = 3/4 - 7/9 = -1/36$.

**(b)** With $a = 9/7$, $r = 3/7$, geometric, $\sum\limits_{k=1}^{\infty} 7^{-k}3^{k+1} = \dfrac{9/7}{1-(3/7)} = 9/4$; with $a = 4/5, r = 2/5$, geometric, $\sum\limits_{k=1}^{\infty}\dfrac{2^{k+1}}{5^k} = \dfrac{4/5}{1-(2/5)} = 4/3$; $\sum\limits_{k=1}^{\infty}\left[7^{-k}3^{k+1} - \dfrac{2^{k+1}}{5^k}\right] = 9/4 - 4/3 = 11/12$.

**3.** **(a)** $p=3>1$, converges. **(b)** $p=1/2 \le 1$, diverges. **(c)** $p=1 \le 1$, diverges. **(d)** $p=2/3 \le 1$, diverges.

**4.** **(a)** $p=4/3>1$, converges. **(b)** $p=1/4 \le 1$, diverges. **(c)** $p=5/3>1$, converges. **(d)** $p=\pi>1$, converges.

**5.** **(a)** $\lim\limits_{k\to+\infty}\dfrac{k^2+k+3}{2k^2+1} = \dfrac{1}{2} \ne 0$; the series diverges. **(b)** $\lim\limits_{k\to+\infty}\left(1+\dfrac{1}{k}\right)^k = e \ne 0$; the series diverges.

**(c)** $\lim\limits_{k\to+\infty}\cos k\pi$ does not exist; the series diverges. **(d)** $\lim\limits_{k\to+\infty}\dfrac{1}{k!} = 0$; no information.



**6.** **(a)** $\lim\limits_{k\to+\infty}\dfrac{k}{e^k} = 0$; no information. **(b)** $\lim\limits_{k\to+\infty}\ln k = +\infty \ne 0$; the series diverges.

**(c)** $\lim\limits_{k\to+\infty}\dfrac{1}{\sqrt{k}} = 0$; no information. **(d)** $\lim\limits_{k\to+\infty}\dfrac{\sqrt{k}}{\sqrt{k}+3} = 1 \ne 0$; the series diverges.

**7.** **(a)** $\displaystyle\int_1^{+\infty}\frac{1}{5x+2} = \lim_{\ell\to+\infty}\frac{1}{5}\ln(5x+2)\Big]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**(b)** $\displaystyle\int_1^{+\infty}\frac{1}{1+9x^2}dx = \lim_{\ell\to+\infty}\frac{1}{3}\tan^{-1}3x\Big]_1^{\ell} = \frac{1}{3}\left(\pi/2 - \tan^{-1}3\right)$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**8.** **(a)** $\displaystyle\int_1^{+\infty}\frac{x}{1+x^2}dx = \lim_{\ell\to+\infty}\frac{1}{2}\ln(1+x^2)\Big]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**(b)** $\displaystyle\int_1^{+\infty}(4+2x)^{-3/2}dx = \lim_{\ell\to+\infty} -1/\sqrt{4+2x}\Big]_1^{\ell} = 1/\sqrt{6}$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**9.** $\sum\limits_{k=1}^{\infty}\dfrac{1}{k+6} = \sum\limits_{k=7}^{\infty}\dfrac{1}{k}$, diverges because the harmonic series diverges.

**10.** $\sum\limits_{k=1}^{\infty}\dfrac{3}{5k} = \sum\limits_{k=1}^{\infty}\dfrac{3}{5}\left(\dfrac{1}{k}\right)$, diverges because the harmonic series diverges.

**11.** $\sum\limits_{k=1}^{\infty}\dfrac{1}{\sqrt{k+5}} = \sum\limits_{k=6}^{\infty}\dfrac{1}{\sqrt{k}}$, diverges because the $p$-series with $p = 1/2 \le 1$ diverges.

**12.** $\lim\limits_{k\to+\infty}\dfrac{1}{e^{1/k}} = 1$, the series diverges by the Divergence Test, because $\lim\limits_{k\to+\infty} u_k = 1 \ne 0$.

**13.** $\displaystyle\int_1^{+\infty}(2x-1)^{-1/3}dx = \lim_{\ell\to+\infty}\frac{3}{4}(2x-1)^{2/3}\bigg]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**14.** $\dfrac{\ln x}{x}$ is decreasing for $x \ge e$, and $\displaystyle\int_3^{+\infty}\frac{\ln x}{x} = \lim_{\ell\to+\infty}\frac{1}{2}(\ln x)^2\bigg]_3^{\ell} = +\infty$, so the series diverges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**15.** $\displaystyle\lim_{k\to+\infty}\frac{k}{\ln(k+1)} = \lim_{k\to+\infty}\frac{1}{1/(k+1)} = +\infty$, the series diverges by the Divergence Test, because $\displaystyle\lim_{k\to+\infty}u_k \neq 0$.

**16.** $\displaystyle\int_1^{+\infty}xe^{-x^2}dx = \lim_{\ell\to+\infty} -\frac{1}{2}e^{-x^2}\bigg]_1^{\ell} = e^{-1}/2$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**17.** $\displaystyle\lim_{k\to+\infty}(1+1/k)^{-k} = 1/e \neq 0$, the series diverges by the Divergence Test.



**18.** $\displaystyle\lim_{k\to+\infty}\frac{k^2+1}{k^2+3} = 1 \neq 0$, the series diverges by the Divergence Test.

**19.** $\displaystyle\int_1^{+\infty}\frac{\tan^{-1}x}{1+x^2}dx = \lim_{\ell\to+\infty}\frac{1}{2}\left(\tan^{-1}x\right)^2\bigg]_1^{\ell} = 3\pi^2/32$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous), since $\dfrac{d}{dx}\dfrac{\tan^{-1}x}{1+x^2} = \dfrac{1-2x\tan^{-1}x}{(1+x^2)^2} < 0$ for $x \ge 1$.

**20.** $\displaystyle\int_1^{+\infty}\frac{1}{\sqrt{x^2+1}}dx = \lim_{\ell\to+\infty}\sinh^{-1}x\bigg]_1^{\ell} = +\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**21.** $\displaystyle\lim_{k\to+\infty}k^2\sin^2(1/k) = 1 \neq 0$, the series diverges by the Divergence Test.

**22.** $\displaystyle\int_1^{+\infty}x^2e^{-x^3}dx = \lim_{\ell\to+\infty} -\frac{1}{3}e^{-x^3}\bigg]_1^{\ell} = e^{-1}/3$, the series converges by the Integral Test (which can be applied, because $x^2e^{-x^3}$ is decreasing for $x \ge 1$, it is continuous and the series has positive terms).

**23.** $7\displaystyle\sum_{k=5}^{\infty}k^{-1.01}$, $p$-series with $p = 1.01 > 1$, converges.

**24.** $\displaystyle\int_1^{+\infty}\operatorname{sech}^2 x\,dx = \lim_{\ell\to+\infty}\tanh x\bigg]_1^{\ell} = 1 - \tanh(1)$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and $f$ is decreasing and continuous).

**25.** $\dfrac{1}{x(\ln x)^p}$ is decreasing for $x \ge e^{-p}$, so use the Integral Test (which can be applied, because $f$ is continuous and the series has positive terms) with $a = e^{\alpha}$, i.e. $\displaystyle\int_{e^{\alpha}}^{+\infty}\frac{dx}{x(\ln x)^p}$ to get $\displaystyle\lim_{\ell\to+\infty}\ln(\ln x)\bigg]_{e^{\alpha}}^{\ell} = +\infty$ if $p = 1$, $\displaystyle\lim_{\ell\to+\infty}\frac{(\ln x)^{1-p}}{1-p}\bigg]_{e^{\alpha}}^{\ell} = \begin{cases} +\infty & \text{if } p < 1 \\ \dfrac{\alpha^{1-p}}{p-1} & \text{if } p > 1 \end{cases}$. Thus the series converges for $p > 1$.

**26.** If $p > 0$ set $g(x) = x(\ln x)[\ln(\ln x)]^p, g'(x) = (\ln(\ln x))^{p-1}\left[(1+\ln x)\ln(\ln x) + p\right]$, and, for $x > e^e$, $g'(x) > 0$, thus $1/g(x)$ is decreasing for $x > e^e$; use the Integral Test with $\displaystyle\int_{e^e}^{+\infty}\frac{dx}{x(\ln x)[\ln(\ln x)]^p}$ to get $\displaystyle\lim_{\ell\to+\infty}\ln[\ln(\ln x)]\bigg]_{e^e}^{\ell} = +\infty$ if $p = 1$, $\displaystyle\lim_{\ell\to+\infty}\frac{[\ln(\ln x)]^{1-p}}{1-p}\bigg]_{e^e}^{\ell} = \begin{cases} +\infty & \text{if } p < 1, \\ \dfrac{1}{p-1} & \text{if } p > 1 \end{cases}$. Thus the series converges for $p > 1$ and diverges for $0 < p \le 1$. If $p \le 0$ then $\dfrac{[\ln(\ln x)]^{-p}}{x\ln x} \ge \dfrac{1}{x\ln x}$ for $x > e^e$ so the series diverges, since $\displaystyle\int\frac{1}{x\ln x}dx$ is divergent by Exercise 25. (The Integral Test can be applied, because $f$ is continuous and the series has positive terms).

**27.** Suppose $\sum(u_k+v_k)$ converges; then so does $\sum[(u_k+v_k)-u_k]$, but $\sum[(u_k+v_k)-u_k] = \sum v_k$, so $\sum v_k$ converges which contradicts the assumption that $\sum v_k$ diverges. Suppose $\sum(u_k-v_k)$ converges; then so does $\sum[u_k-(u_k-v_k)] = \sum v_k$ which leads to the same contradiction as before.

**28.** Let $u_k = 2/k$ and $v_k = 1/k$; then both $\sum(u_k+v_k)$ and $\sum(u_k-v_k)$ diverge; let $u_k = 1/k$ and $v_k = -1/k$ then $\sum(u_k+v_k)$ converges; let $u_k = v_k = 1/k$ then $\sum(u_k-v_k)$ converges.



**29.** **(a)** Diverges because $\displaystyle\sum_{k=1}^{\infty}(2/3)^{k-1}$ converges (geometric series, $r = 2/3$, $|r| < 1$) and $\displaystyle\sum_{k=1}^{\infty}1/k$ diverges (the harmonic series).

**(b)** Diverges because $\displaystyle\sum_{k=1}^{\infty}1/(3k+2)$ diverges (Integral Test) and $\displaystyle\sum_{k=1}^{\infty}1/k^{3/2}$ converges ($p$-series, $p = 3/2 > 1$).

**30.** **(a)** Converges because both $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^2}$ (Exercise 25) and $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ converge ($p$-series, $p = 2 > 1$).

**(b)** Diverges, because $\displaystyle\sum^{+\infty}ke^{-k^2}$ converges (Integral Test), and, by Exercise 25, $\displaystyle\sum^{+\infty}\frac{1}{k\ln k}$ diverges.

[redacted]

**31.** False; if $\sum u_k$ converges then $\lim u_k = 0$, so $\lim \frac{1}{u_k}$ diverges, so $\sum \frac{1}{u_k}$ cannot converge.

**32.** True; if $\sum cu_k$ diverges then $c \neq 0$ so $\sum u_k$ diverges.

**33.** True, see Theorem 9.4.4.

**34.** False, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is a $p$-series.

**35.** **(a)** $3\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{k^4} = \pi^2/2 - \pi^4/90$. **(b)** $\sum_{k=1}^{\infty} \frac{1}{k^2} - 1 - \frac{1}{2^2} = \pi^2/6 - 5/4$. **(c)** $\sum_{k=2}^{\infty} \frac{1}{(k-1)^4} = \sum_{k=1}^{\infty} \frac{1}{k^4} = \pi^4/90$.

**36.** **(a)** If $S = \sum_{k=1}^{\infty} u_k$ and $s_n = \sum_{k=1}^{n} u_k$, then $S - s_n = \sum_{k=n+1}^{\infty} u_k$. Interpret $u_k$, $k = n+1, n+2, \ldots$ as the areas of inscribed or circumscribed rectangles with height $u_k$ and base of length one for the curve $y = f(x)$ to obtain the result.

**(b)** Add $s_n = \sum_{k=1}^{n} u_k$ to each term in the conclusion of part (a) to get the desired result: $s_n + \int_{n+1}^{+\infty} f(x)\,dx < \sum_{k=1}^{+\infty} u_k < s_n + \int_{n}^{+\infty} f(x)\,dx$.

**37.** **(a)** In Exercise 36 above let $f(x) = \frac{1}{x^2}$. Then $\int_{n}^{+\infty} f(x)\,dx = -\frac{1}{x}\Big]_{n}^{+\infty} = \frac{1}{n}$; use this result and the same result with $n+1$ replacing $n$ to obtain the desired result.

**(b)** $s_3 = 1 + 1/4 + 1/9 = 49/36$; $58/36 = s_3 + \frac{1}{4} < \frac{1}{6}\pi^2 < s_3 + \frac{1}{3} = 61/36$.

**(d)** $1/11 < \frac{1}{6}\pi^2 - s_{10} < 1/10$.

**38.** Apply Exercise 36 in each case:

**(a)** $f(x) = \frac{1}{(2x+1)^2}$, $\int_{n}^{+\infty} f(x)\,dx = \frac{1}{2(2n+1)}$, so $\frac{1}{46} < \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} - s_{10} < \frac{1}{42}$.



**(b)** $f(x) = \frac{1}{k^2+1}$, $\int_{n}^{+\infty} f(x)\,dx = \frac{\pi}{2} - \tan^{-1}(n)$, so $\pi/2 - \tan^{-1}(11) < \sum_{k=1}^{\infty} \frac{1}{k^2+1} - s_{10} < \pi/2 - \tan^{-1}(10)$.

**(c)** $f(x) = \frac{x}{e^x}$, $\int_{n}^{+\infty} f(x)\,dx = (n+1)e^{-n}$, so $12e^{-11} < \sum_{k=1}^{\infty} \frac{k}{e^k} - s_{10} < 11e^{-10}$.

**39.** **(a)** Let $S_n = \sum_{k=1}^{n} \frac{1}{k^4}$ By Exercise 36(a), with $f(x) = \frac{1}{x^4}$, the result follows.

**(b)** $h(x) = \frac{1}{3x^3} - \frac{1}{3(x+1)^3}$ is a decreasing function, and the smallest $n$ such that $\left| \frac{1}{3n^3} - \frac{1}{3(n+1)^3} \right| \leq 0.001$ is $n = 6$.

**(c)** The midpoint of the interval indicated in Part c is $S_6 + \dfrac{\frac{1}{3 \cdot 6^3} + \frac{1}{3 \cdot 7^3}}{2} \approx 1.082381$. A calculator gives $\pi^4/90 \approx 1.08232$.

**40.** **(a)** Let $F(x) = \frac{1}{x}$, then $\int_{1}^{n} \frac{1}{x}\,dx = \ln n$ and $\int_{1}^{n+1} \frac{1}{x}\,dx = \ln(n+1)$, $u_1 = 1$, so $\ln(n+1) < s_n < 1 + \ln n$.

**(b)** $\ln(1{,}000{,}001) < s_{1{,}000{,}000} < 1 + \ln(1{,}000{,}000)$, $13 < s_{1{,}000{,}000} < 15$.

**(c)** $s_{10^9} < 1 + \ln 10^9 = 1 + 9\ln 10 < 22$.

**(d)** $s_n > \ln(n+1) \geq 100$, $n \geq e^{100} - 1 \approx 2.688 \times 10^{43}$; $n = 2.69 \times 10^{43}$.

**41.** $x^2e^{-x}$ is continuous, decreasing and positive for $x > 2$ so the Integral Test applies: $\int_{1}^{\infty} x^2e^{-x}\,dx = -(x^2 + 2x + 2)e^{-x}\Big]_{1}^{\infty} = 5e^{-1}$ so the series converges.

**42.** **(a)** $f(x) = 1/(x^3+1)$ is continuous, decreasing and positive on the interval $[1, +\infty]$, so the Integral Test applies.

**(c)**

| $n$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s_n$ | 0.681980 | 0.685314 | 0.685966 | 0.686199 | 0.686307 | 0.686367 | 0.686403 | 0.686426 | 0.686442 | 0.686454 |

**(e)** Set $g(n) = \int_{n}^{+\infty} \frac{1}{x^3+1}\,dx = \frac{\sqrt{3}}{6}\pi + \frac{1}{6}\ln\frac{n^3+1}{(n+1)^3} - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2n-1}{\sqrt{3}}\right)$; for $n \geq 13$, $g(n) - g(n+1) \leq 0.0005$; $s_{13} + (g(13) + g(14))/2 \approx 0.6865$, so the sum $\approx 0.6865$ to three decimal places.

## Exercise Set 9.5

All convergence tests in this section require that the series have positive terms - this requirement is met in all these exercises.

**1.** **(a)** $\frac{1}{5k^2 - k} \leq \frac{1}{5k^2 - k^2} = \frac{1}{4k^2}$, $\sum_{k=1}^{\infty} \frac{1}{4k^2}$ converges, so the original series also converges.

**(b)** $\frac{3}{k - 1/4} > \frac{3}{k}$, $\sum_{k=1}^{\infty} \frac{3}{k}$ diverges, so the original series also diverges.

**2.** (a) $\dfrac{k+1}{k^2-k} > \dfrac{k}{k^2} = \dfrac{1}{k}$, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, so the original series also diverges.

(b) $\dfrac{2}{k^4+k} < \dfrac{2}{k^4}$, $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^4}$ converges, so the original series also converges.

**3.** (a) $\dfrac{1}{3^k+5} < \dfrac{1}{3^k}$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges, so the original series also converges.

(b) $\dfrac{5\sin^2 k}{k!} < \dfrac{5}{k!}$, $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k!}$ converges, so the original series also converges.

**4.** (a) $\dfrac{\ln k}{k} > \dfrac{1}{k}$ for $k \geq 3$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so the original series also diverges.

(b) $\dfrac{k}{k^{3/2}-1/2} > \dfrac{k}{k^{3/2}} = \dfrac{1}{\sqrt{k}}$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, so the original series also diverges.

**5.** Compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{4k^7-2k^6+6k^5}{8k^7+k-8} = 1/2$, which is finite and positive, therefore the original series converges.

**6.** Compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k}{9k+6} = 1/9$, which is finite and positive, therefore the original series diverges.

**7.** Compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{5}{3^k}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{3^k}{3^k+1} = 1$, which is finite and positive, therefore the original series converges.

**8.** Compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k^2(k+3)}{(k+1)(k+2)(k+5)} = 1$, which is finite and positive, therefore the original series diverges.

**9.** Compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k^{2/3}}{(8k^2-3k)^{1/3}} = \lim_{k\to+\infty} \frac{1}{(8-3/k)^{1/3}} = 1/2$, which is finite and positive, therefore the original series diverges.

**10.** Compare with the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{17}}$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k^{17}}{(2k+3)^{17}} = \lim_{k\to+\infty} \frac{1}{(2+3/k)^{17}} = 1/2^{17}$, which is finite and positive, therefore the original series converges.

**11.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{3^{k+1}/(k+1)!}{3^k/k!} = \lim_{k\to+\infty} \frac{3}{k+1} = 0 < 1$, the series converges.

**12.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{4^{k+1}/(k+1)^2}{4^k/k^2} = \lim_{k\to+\infty} \frac{4k^2}{(k+1)^2} = 4 > 1$, the series diverges.

**13.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{k}{k+1} = 1$, the result is inconclusive.



**14.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)(1/2)^{k+1}}{k(1/2)^k} = \lim_{k\to+\infty} \frac{k+1}{2k} = 1/2 < 1$, the series converges.

**15.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)!/(k+1)^3}{k!/k^3} = \lim_{k\to+\infty} \frac{k^3}{(k+1)^2} = +\infty$, the series diverges.

**16.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)/[(k+1)^2+1]}{k/(k^2+1)} = \lim_{k\to+\infty} \frac{(k+1)(k^2+1)}{k(k^2+2k+2)} = 1$, the result is inconclusive.

**17.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{3k+2}{2k-1} = 3/2 > 1$, the series diverges.

**18.** $\rho = \displaystyle\lim_{k\to+\infty} k/100 = +\infty$, the series diverges.

**19.** $\rho = \displaystyle\lim_{k\to+\infty} \frac{k^{1/k}}{5} = 1/5 < 1$, the series converges.

**20.** $\rho = \displaystyle\lim_{k\to+\infty} (1-e^{-k}) = 1$, the result is inconclusive.

**21.** False; it uses terms from two different sequences.

**22.** True, Ratio Test.

**23.** True, Limit Comparison Test with $v_k = 1/k^2$.

**24.** False; it decides convergence based on a limit of $k$-th roots of the terms of the series.

**25.** Ratio Test, $\rho = \displaystyle\lim_{k\to+\infty} 7/(k+1) = 0$, converges.

**26.** Limit Comparison Test, compare with the divergent series $\displaystyle\sum_{k=1}^{\infty} 1/k$, $\rho = \displaystyle\lim_{k\to+\infty} \frac{k}{2k+1} = 1/2$, which is finite and positive, therefore the original series diverges.

**27.** Ratio Test, $\rho = \displaystyle\lim_{k\to+\infty} \frac{(k+1)^2}{5k^2} = 1/5 < 1$, converges.

**28.** Ratio Test, $\rho = \displaystyle\lim_{k\to+\infty} (10/3)(k+1) = +\infty$, diverges.

**29.** Ratio Test, $\rho = \lim_{k\to+\infty} e^{-1}(k+1)^{50}/k^{50} = e^{-1} < 1$, converges

**30.** Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$.

**31.** Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} 1/k^{5/2}$, $\rho = \lim_{k\to+\infty} \frac{k^3}{k^3+1} = 1$, converges

**32.** $\frac{4}{2+3^k k} < \frac{4}{3^k k}$, $\sum_{k=1}^{\infty} \frac{4}{3^k k}$ converges (Ratio Test) so $\sum_{k=1}^{\infty} \frac{4}{2+k3^k}$ converges by the Comparison Test

**33.** Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/k$, $\rho = \lim_{k\to+\infty} \frac{k}{\sqrt{k^2+k}} = 1$, diverges



**34.** $\frac{2+(-1)^k}{5^k} \le \frac{3}{5^k}$, $\sum_{k=1}^{\infty} 3/5^k$ converges so $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{5^k}$ converges by the Comparison Test

**35.** Limit Comparison Test, compare with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$, $\rho = \lim_{k\to+\infty} \frac{k^3+2k^{5/2}}{k^3+3k^2+3k} = 1$, converges

**36.** $\frac{4+\cos k}{k^3} < \frac{5}{k^3}$, $\sum_{k=1}^{\infty} 5/k^3$ converges so $\sum_{k=1}^{\infty} \frac{4+\cos k}{k^3}$ converges

**37.** Limit Comparison Test, compare with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$

**38.** Ratio Test, $\rho = \lim_{k\to+\infty} (1+1/k)^{-k} = 1/e < 1$, converges

**39.** Ratio Test, $\rho = \lim_{k\to+\infty} \frac{\ln(k+1)}{e\ln k} = \lim_{k\to+\infty} \frac{k}{e(k+1)} = 1/e < 1$, converges

**40.** Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+1}{e^{2k+1}} = \lim_{k\to+\infty} \frac{1}{2e^{2k+1}} = 0$, converges

**41.** Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+5}{4(k+1)} = 1/4$, converges

**42.** Root Test, $\rho = \lim_{k\to+\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to+\infty} \frac{1}{(1+1/k)^k} = 1/e$, converges

**43.** Diverges by the Divergence Test, because $\lim_{k\to+\infty} \frac{1}{4+2^{-k}} = 1/4 \neq 0$

**44.** $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1} = \sum_{k=2}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$ because $\ln 1 = 0$, $\frac{\sqrt{k}\ln k}{k^3+1} < \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$, $\int_2^{+\infty} \frac{\ln x}{x^2}\,dx = \lim_{\ell\to+\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big]_2^{\ell} = \frac{1}{2}(\ln 2 + 1)$, so $\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$ converges and so does $\sum_{k=1}^{\infty} \frac{\sqrt{k}\ln k}{k^3+1}$.

**45.** $\frac{\tan^{-1} k}{k^2} < \frac{\pi/2}{k^2}$, $\sum_{k=1}^{\infty} \frac{\pi/2}{k^2}$ converges so $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$ converges.

**46.** $\frac{5^k+k}{k!+3} < \frac{5^k+5^k}{k!} = \frac{2(5^k)}{k!}$, $\sum_{k=1}^{\infty} 2\left(\frac{5^k}{k!}\right)$ converges (Ratio Test), so $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ converges

**47.** Ratio Test, $\rho = \lim_{k\to+\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = 1/4$, converges

**48.** Root Test: $\rho = \lim_{k\to+\infty} \frac{\pi(k+1)}{k^{1+1/k}} = \lim_{k\to+\infty} \pi\frac{k+1}{k} = \pi$, diverges

**49.** $a_k = \frac{\ln k}{3^k}$, $\frac{a_{k+1}}{a_k} = \frac{\ln(k+1)}{\ln k}\frac{3^k}{3^{k+1}} \to \frac{1}{3}$, converges

**50.** $a_k = \frac{\alpha^k}{k^\alpha}$, $\frac{a_{k+1}}{a_k} = \alpha\left(\frac{k+1}{k}\right)^{\alpha} \to \alpha$, converges if and only if $\alpha < 1$. ($\alpha = 1$ harmonic series)



**51.** $u_k = \frac{k!}{1\cdot3\cdot5\cdot\ldots(2k-1)}$, by the Ratio Test $\rho = \lim_{k\to+\infty} \frac{k+1}{2k+1} = 1/2$; converges

**52.** $u_k = \frac{1\cdot3\cdot5\cdot\ldots(2k-1)}{(2k-1)!}$, by the Ratio Test $\rho = \lim_{k\to+\infty} \frac{1}{2k} = 0$; converges

**53.** Set $g(x) = \sqrt{x} - \ln x$; $\frac{d}{dx}g(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = 0$ only at $x = 4$. Since $\lim_{x\to0^+} g(x) = \lim_{x\to+\infty} g(x) = +\infty$ it follows that $g(x)$ has its absolute minimum at $x = 4$, $g(4) = \sqrt{4} - \ln 4 > 0$, and thus $\sqrt{x} - \ln x > 0$ for $x > 0$.

**(a)** $\frac{\ln k}{k^2} < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$, $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges so $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges.

**(b)** $\frac{1}{(\ln k)^2} > \frac{1}{k}$, $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges so $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$ diverges

**54. (b)** $\rho = \lim_{k\to+\infty} \dfrac{\sin(\pi/k)}{\pi/k} = 1$ and $\sum_{k=1}^{\infty} \pi/k$ diverges, so the original series also diverges.

**55. (a)** $\cos x \approx 1 - x^2/2$, $1 - \cos\left(\dfrac{1}{k}\right) \approx \dfrac{1}{2k^2}$. **(b)** $\rho = \lim_{k\to+\infty} \dfrac{1-\cos(1/k)}{1/k^2} = 1/2$, converges.

**56. (a)** If $\lim_{k\to+\infty} (a_k/b_k) = 0$ then for $k \geq K$, $a_k/b_k < 1$, $a_k < b_k$ so $\sum a_k$ converges by the Comparison Test.

**(b)** If $\lim_{k\to+\infty} (a_k/b_k) = +\infty$ then for $k \geq K$, $a_k/b_k > 1$, $a_k > b_k$ so $\sum a_k$ diverges by the Comparison Test.

**57. (a)** If $\sum b_k$ converges, then set $M = \sum b_k$. Then $a_1 + a_2 + \ldots + a_n \leq b_1 + b_2 + \ldots + b_n \leq M$; apply Theorem 9.4.6 to get convergence of $\sum a_k$.

**(b)** Assume the contrary, that $\sum b_k$ converges; then use part (a) of the Theorem to show that $\sum a_k$ converges, a contradiction.

## Exercise Set 9.7

**1.** **(a)** $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(0) = (-1)^k$; $e^{-x} \approx 1 - x + x^2/2$ (quadratic), $e^{-x} \approx 1 - x$ (linear).

**(b)** $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $\cos x \approx 1 - x^2/2$ (quadratic), $\cos x \approx 1$ (linear).

**2.** **(a)** $f'(x) = \cos x$, $f''(x) = -\sin x$, $f(\pi/2) = 1$, $f'(\pi/2) = 0$, $f''(\pi/2) = -1$, $\sin x \approx 1 - (x - \pi/2)^2/2$ (quadratic), $\sin x \approx 1$ (linear).

**(b)** $f(1) = 1$, $f'(1) = 1/2$, $f''(1) = -1/4$; $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$ (quadratic), $\sqrt{x} \approx 1 + \frac{1}{2}(x-1)$ (linear).

**3.** **(a)** $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$; $f(1) = 1, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{4}$; $\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$.

**(b)** $x = 1.1$, $x_0 = 1$, $\sqrt{1.1} \approx 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1.04875$, calculator value $\approx 1.0488088$.

**4.** **(a)** $\cos x \approx 1 - x^2/2$.

**(b)** $2^\circ = \pi/90$ rad, $\cos 2^\circ = \cos(\pi/90) \approx 1 - \dfrac{\pi^2}{2 \cdot 90^2} \approx 0.99939077$, calculator value $\approx 0.99939083$.

**5.** $f(x) = \tan x$, $61^\circ = \pi/3 + \pi/180$ rad; $x_0 = \pi/3$, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$; $f(\pi/3) = \sqrt{3}, f'(\pi/3) = 4, f''(x) = 8\sqrt{3}$; $\tan x \approx \sqrt{3} + 4(x - \pi/3) + 4\sqrt{3}(x - \pi/3)^2$, $\tan 61^\circ = \tan(\pi/3 + \pi/180) \approx \sqrt{3} + 4\pi/180 + 4\sqrt{3}(\pi/180)^2 \approx 1.80397443$, calculator value $\approx 1.80404776$.

**6.** $f(x) = \sqrt{x}$, $x_0 = 36, f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$; $f(36) = 6, f'(36) = \frac{1}{12}, f''(36) = -\frac{1}{864}$; $\sqrt{x} \approx 6 + \frac{1}{12}(x-36) - \frac{1}{1728}(x-36)^2$; $\sqrt{36.03} \approx 6 + \frac{0.03}{12} - \frac{(0.03)^2}{1728} \approx 6.00249947917$, calculator value $\approx 6.00249947938$.

**7.** $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(0) = (-1)^k; p_0(x) = 1$, $p_1(x) = 1 - x$, $p_2(x) = 1 - x + \frac{1}{2}x^2$, $p_3(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3$, $p_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$; $\sum_{k=0}^{n} \frac{(-1)^k}{k!}x^k$.

**8.** $f^{(k)}(x) = a^k e^{ax}$, $f^{(k)}(0) = a^k$; $p_0(x) = 1$, $p_1(x) = 1 + ax$, $p_2(x) = 1 + ax + \frac{a^2}{2}x^2$, $p_3(x) = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3$, $p_4(x) = 1 + ax + \frac{a^2}{2}x^2 + \frac{a^3}{3!}x^3 + \frac{a^4}{4!}x^4$; $\sum_{k=0}^{n} \frac{a^k}{k!}x^k$.

**9.** $f^{(k)}(0) = 0$ if $k$ is odd, $f^{(k)}(0)$ is alternately $\pi^k$ and $-\pi^k$ if $k$ is even; $p_0(x) = 1$, $p_1(x) = 1$, $p_2(x) = 1 - \frac{\pi^2}{2!}x^2$; $p_3(x) = 1 - \frac{\pi^2}{2!}x^2$, $p_4(x) = 1 - \frac{\pi^2}{2!}x^2 + \frac{\pi^4}{4!}x^4$; $\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \pi^{2k}}{(2k)!}x^{2k}$.

NB: The function $[x]$ defined for real $x$ indicates the greatest integer which is $\leq x$.

**10.** $f^{(k)}(0) = 0$ if $k$ is even, $f^{(k)}(0)$ is alternately $\pi^k$ and $-\pi^k$ if $k$ is odd; $p_0(x) = 0$, $p_1(x) = \pi x$, $p_2(x) = \pi x$; $p_3(x) = \pi x - \frac{\pi^3}{3!}x^3$, $p_4(x) = \pi x - \frac{\pi^3}{3!}x^3$; $\sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!}x^{2k+1}$.

NB: If $n = 0$ then $[\frac{n-1}{2}] = -1$; by definition any sum which runs from $k = 0$ to $k = -1$ is called the 'empty sum' and has value 0.



**11.** $f^{(0)}(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k+1}(k-1)!}{(1+x)^k}$, $f^{(k)}(0) = (-1)^{k+1}(k-1)!$; $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x - \frac{1}{2}x^2$, $p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$, $p_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$; $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}x^k$.

**12.** $f^{(k)}(x) = (-1)^k \dfrac{k!}{(1+x)^{k+1}}$; $f^{(k)}(0) = (-1)^k k!$; $p_0(x) = 1$, $p_1(x) = 1 - x$, $p_2(x) = 1 - x + x^2$, $p_3(x) = 1 - x + x^2 - x^3$, $p_4(x) = 1 - x + x^2 - x^3 + x^4$; $\sum_{k=0}^{n} (-1)^k x^k$.

**13.** $f^{(k)}(0) = 0$ if $k$ is odd, $f^{(k)}(0) = 1$ if $k$ is even; $p_0(x) = 1, p_1(x) = 1$, $p_2(x) = 1 + x^2/2$, $p_3(x) = 1 + x^2/2$, $p_4(x) = 1 + x^2/2 + x^4/4!$; $\sum_{k=0}^{[\frac{n}{2}]} \frac{1}{(2k)!}x^{2k}$.

**14.** $f^{(k)}(0) = 0$ if $k$ is even, $f^{(k)}(0) = 1$ if $k$ is odd; $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$, $p_3(x) = x + x^3/3!$, $p_4(x) = x + x^3/3!$; $\sum_{k=0}^{[\frac{n-1}{2}]} \frac{1}{(2k+1)!}x^{2k+1}$.

**15.** $f^{(k)}(x) = \begin{cases} (-1)^{k/2}(x \sin x - k \cos x) & k \text{ even} \\ (-1)^{(k-1)/2}(x \cos x + k \sin x) & k \text{ odd} \end{cases}$, $f^{(k)}(0) = \begin{cases} (-1)^{1+k/2}k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$. $p_0(x) = 0$, $p_1(x) = 0$, $p_2(x) = x^2$, $p_3(x) = x^2$, $p_4(x) = x^2 - \frac{1}{6}x^4$; $\sum_{k=0}^{[\frac{n}{2}]-1} \frac{(-1)^k}{(2k+1)!}x^{2k+2}$.

**16.** $f^{(k)}(x) = (k + x)e^x$, $f^{(k)}(0) = k$; $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x + x^2$, $p_3(x) = x + x^2 + \frac{1}{2}x^3$, $p_4(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3!}x^4$; $\sum_{k=1}^{n} \frac{1}{(k-1)!}x^k$.

**17.** $f^{(k)}(x_0) = e$; $p_0(x) = e$, $p_1(x) = e + e(x-1)$, $p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$, $p_3(x) = e + e(x-1) + \frac{e}{2}(x -$

**17.** $f^{(k)}(x_0) = e$; $p_0(x) = e$, $p_1(x) = e + e(x-1)$, $p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$, $p_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3$, $p_4(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4$; $\sum_{k=0}^{n} \frac{e}{k!}(x-1)^k$.

**18.** $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(\ln 2) = (-1)^k \frac{1}{2}$; $p_0(x) = \frac{1}{2}$, $p_1(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2)$, $p_2(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2) + \frac{1}{2 \cdot 2}(x - \ln 2)^2$, $p_3(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2) + \frac{1}{2 \cdot 2}(x - \ln 2)^2 - \frac{1}{2 \cdot 3!}(x - \ln 2)^3$, $p_4(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2) + \frac{1}{2 \cdot 2}(x - \ln 2)^2 - \frac{1}{2 \cdot 3!}(x - \ln 2)^3 + \frac{1}{2 \cdot 4!}(x - \ln 2)^4$; $\sum_{k=0}^{n} \frac{(-1)^k}{2 \cdot k!}(x - \ln 2)^k$.

**19.** $f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$, $f^{(k)}(-1) = -k!$; $p_0(x) = -1$; $p_1(x) = -1 - (x+1)$, $p_2(x) = -1 - (x+1) - (x+1)^2$; $p_3(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3$; $p_4(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3 - (x+1)^4$; $\sum_{k=0}^{n} (-1)(x+1)^k$.

**20.** $f^{(k)}(x) = \frac{(-1)^k k!}{(x+2)^{k+1}}$, $f^{(k)}(3) = \frac{(-1)^k k!}{5^{k+1}}$; $p_0(x) = \frac{1}{5}$; $p_1(x) = \frac{1}{5} - \frac{1}{25}(x-3)$; $p_2(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2$; $p_3(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3$; $p_4(x) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3 + \frac{1}{3125}(x-3)^4$; $\sum_{k=0}^{n} \frac{(-1)^k}{5^{k+1}}(x-3)^k$.



**21.** $f^{(k)}(1/2) = 0$ if $k$ is odd, $f^{(k)}(1/2)$ is alternately $\pi^k$ and $-\pi^k$ if $k$ is even; $p_0(x) = p_1(x) = 1, p_2(x) = p_3(x) = 1 - \frac{\pi^2}{2}(x-1/2)^2$. $p_4(x) = 1 - \frac{\pi^2}{2}(x-1/2)^2 + \frac{\pi^4}{4!}(x-1/2)^4$; $\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k \pi^{2k}}{(2k)!}(x-1/2)^{2k}$.

**22.** $f^{(k)}(\pi/2) = 0$ if $k$ is even, $f^{(k)}(\pi/2)$ is alternately $-1$ and $1$ if $k$ is odd; $p_0(x) = 0$, $p_1(x) = -(x - \pi/2)$, $p_2(x) = -(x-\pi/2)$, $p_3(x) = -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3$, $p_4(x) = -(x-\pi/2) + \frac{1}{3!}(x-\pi/2)^3$; $\sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi/2)^{2k+1}$.

**23.** $f(1) = 0$, for $k \geq 1$, $f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(1) = (-1)^{k-1}(k-1)!$; $p_0(x) = 0$, $p_1(x) = (x-1)$; $p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$; $p_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, $p_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$; $\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$.

**24.** $f(e) = 1$, for $k \geq 1$, $f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k}$; $f^{(k)}(e) = \frac{(-1)^{k-1}(k-1)!}{e^k}$; $p_0(x) = 1$, $p_1(x) = 1 + \frac{1}{e}(x-e)$; $p_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2$; $p_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$, $p_4(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 - \frac{1}{4e^4}(x-e)^4$; $1 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{ke^k}(x-e)^k$.

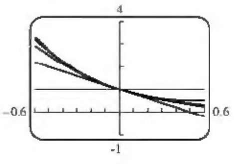
**25.** **(a)** $f(0) = 1, f'(0) = 2, f''(0) = -2, f'''(0) = 6$, the third MacLaurin polynomial for $f(x)$ is $f(x)$.

**(b)** $f(1) = 1, f'(1) = 2, f''(1) = -2, f'''(1) = 6$, the third Taylor polynomial for $f(x)$ is $f(x)$.

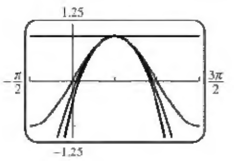
**26.** **(a)** $f^{(k)}(0) = k!c_k$ for $k \leq n$; the $n$th Maclaurin polynomial for $f(x)$ is $f(x)$.

**(b)** $f^{(k)}(x_0) = k!c_k$ for $k \leq n$; the $n$th Taylor polynomial about $x = 1$ for $f(x)$ is $f(x)$.

**27.** $f^{(k)}(0) = (-2)^k$; $p_0(x) = 1$, $p_1(x) = 1 - 2x$, $p_2(x) = 1 - 2x + 2x^2$, $p_3(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3$.



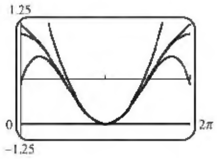

**28.** $f^{(k)}(\pi/2) = 0$ if $k$ is odd, $f^{(k)}(\pi/2)$ is alternately $1$ and $-1$ if $k$ is even; $p_0(x) = 1$, $p_2(x) = 1 - \frac{1}{2}(x-\pi/2)^2$, $p_4(x) = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{24}(x-\pi/2)^4$, $p_6(x) = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{24}(x-\pi/2)^4 - \frac{1}{720}(x-\pi/2)^6$.



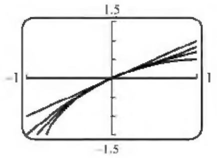



**29.** $f^{(k)}(\pi) = 0$ if $k$ is odd, $f^{(k)}(\pi)$ is alternately $-1$ and $1$ if $k$ is even; $p_0(x) = -1$, $p_2(x) = -1 + \frac{1}{2}(x-\pi)^2$, $p_4(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$, $p_6(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \frac{1}{720}(x-\pi)^6$.

**29.** $f^{(k)}(\pi) = 0$ if $k$ is odd, $f^{(k)}(\pi)$ is alternately $-1$ and $1$ if $k$ is even; $p_0(x) = -1$, $p_2(x) = -1 + \dfrac{1}{2}(x-\pi)^2$, $p_4(x) = -1 + \dfrac{1}{2}(x-\pi)^2 - \dfrac{1}{24}(x-\pi)^4$, $p_6(x) = -1 + \dfrac{1}{2}(x-\pi)^2 - \dfrac{1}{24}(x-\pi)^4 + \dfrac{1}{720}(x-\pi)^6$.




**30.** $f(0) = 0$; for $k \geq 1$, $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{(x+1)^k}$, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$; $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x - \dfrac{1}{2}x^2$, $p_3(x) = x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3$.




**31.** True.

**32.** True, $a_0 = f(0)$.

**33.** False, $p_6^{(4)}(x_0) = f^{(4)}(x_0)$.

**34.** False, since $M = e^2$, $|e^2 - p_4(2)| \leq \dfrac{M|x-0|^{n+1}}{(n+1)!} \leq \dfrac{e^2 \cdot 2^5}{5!} < \dfrac{9 \cdot 2^5}{5!}$.

**35.** $\sqrt{e} = e^{1/2}$, $f(x) = e^x$, $M = e^{1/2}$, $|e^{1/2} - p_n(1/2)| \leq M\dfrac{|x-1/2|^{n+1}}{(n+1)!} \leq 0.00005$, by experimentation take $n = 5$, $\sqrt{e} \approx p_5(1/2) \approx 1.648698$, calculator value $\approx 1.648721$, difference $\approx 0.000023$.

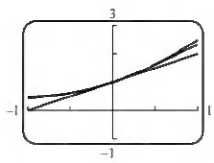
**36.** $1/e = e^{-1}$, $f(x) = e^x$, $M_n = \max|f^{(n+1)}(x)| = e^0 = 1$, $|e^{-1} - p_n(-1)| \leq M\dfrac{|0+1|^{n+1}}{(n+1)!}$, so want $\dfrac{1}{(n+1)!} \leq 0.0005$, $n = 7$, $e^{-1} \approx p_7(-1) \approx 0.367857$, calculator gives $e^{-1} \approx 0.367879$, $|1/e - p_7(-1)| \approx 0.000022$.

**37.** $p(0) = 1$, $p(x)$ has slope $-1$ at $x = 0$, and $p(x)$ is concave up at $x = 0$, eliminating I, II and III respectively and leaving IV.

**38.** Let $p_0(x) = 2$, $p_1(x) = p_2(x) = 2 - 3(x-1)$, $p_3(x) = 2 - 3(x-1) + (x-1)^3$.

**39.** From Exercise 2(a), $p_1(x) = 1 + x$, $p_2(x) = 1 + x + x^2/2$.



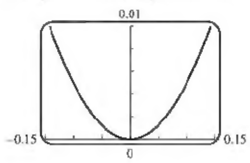



**(a)**

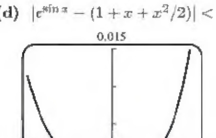
**(b)**

| $x$ | $-1.000$ | $-0.750$ | $-0.500$ | $-0.250$ | $0.000$ | $0.250$ | $0.500$ | $0.750$ | $1.000$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.431 | 0.506 | 0.619 | 0.781 | 1.000 | 1.281 | 1.615 | 1.977 | 2.320 |
| $p_1(x)$ | 0.000 | 0.250 | 0.500 | 0.750 | 1.000 | 1.250 | 1.500 | 1.750 | 2.000 |
| $p_2(x)$ | 0.500 | 0.531 | 0.625 | 0.781 | 1.000 | 1.281 | 1.625 | 2.031 | 2.500 |

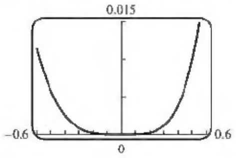
**(c)** $|e^{\sin x} - (1+x)| < 0.01$ for $-0.14 < x < 0.14$.




**(d)** $|e^{\sin x} - (1 + x + x^2/2)| < 0.01$ for $-0.50 < x < 0.50$.



**(d)** $|e^{\sin x} - (1 + x + x^2/2)| < 0.01$ for $-0.50 < x < 0.50$.

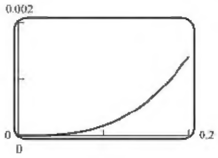



**40.** **(a)** $\cos\alpha \approx 1 - \alpha^2/2$; $x = r - r\cos\alpha = r(1 - \cos\alpha) \approx r\alpha^2/2$.

**(b)** In Figure Ex-36 let $r = 4000$ mi and $\alpha = 1/80$ so that the arc has length $2r\alpha = 100$ mi. Then $x \approx r\alpha^2/2 = \dfrac{4000}{2 \cdot 80^2} = 5/16$ mi.

**41.** **(a)** $f^{(k)}(x) = e^x \le e^b$, $|R_2(x)| \le \dfrac{e^b b^3}{3!} < 0.0005$, $e^b b^3 < 0.003$ if $b \le 0.137$ (by trial and error with a hand calculator), so $[0, 0.137]$.

**(b)**

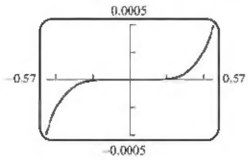



**42.** $f^{(k)}(\ln 4) = 15/8$ for $k$ even, $f^{(k)}(\ln 4) = 17/8$ for $k$ odd, which can be written as $f^{(k)}(\ln 4) = \dfrac{16 - (-1)^k}{8}$;

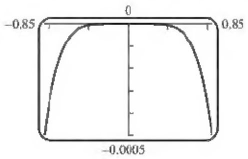


$$\sum_{k=0}^{n} \frac{16 - (-1)^k}{8k!}(x - \ln 4)^k.$$

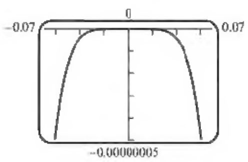
**43.** $\sin x = x - \dfrac{x^3}{3!} + 0 \cdot x^4 + R_4(x)$, $|R_4(x)| \le \dfrac{|x|^5}{5!} < 0.5 \times 10^{-3}$ if $|x|^5 < 0.06$, $|x| < (0.06)^{1/5} \approx 0.569$, $(-0.569, 0.569)$.



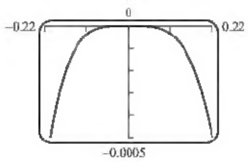

**44.** $M = 1$, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + R_5(x)$, $R_5(x) \le \dfrac{1}{6!}|x|^6 \le 0.0005$ if $|x| < 0.8434$.




**45.** $f^{(6)}(x) = \dfrac{46080x^6}{(1+x^2)^7} - \dfrac{57600x^4}{(1+x^2)^6} + \dfrac{17280x^2}{(1+x^2)^5} - \dfrac{720}{(1+x^2)^4}$, assume first that $|x| < 1/2$, then $|f^{(6)}(x)| < 46080|x|^6 + 57600|x|^4 + 17280|x|^2 + 720$, so let $M = 9360$, $R_5(x) \le \dfrac{9360}{5!}|x|^5 < 0.0005$ if $x < 0.0915$.




**46.** $f(x) = \ln(1 + x)$, $f^{(4)}(x) = -6/(1 + x)^4$, first assume $|x| < 0.8$, then we can calculate $M = 6/2^{-4} = 96$, and $|f(x) - p(x)| \le \dfrac{96}{4!}|x|^4 < 0.0005$ if $|x| < 0.1057$.




## Exercise Set 9.8

**1.** $f^{(k)}(x) = (-1)^k e^{-x}$, $f^{(k)}(0) = (-1)^k$; $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$.